THE ARITHMETIC PERFECTION OF
BUCHSBAUM-EISENBUD VARIETIES
AND GENERIC MODULES OF PROJECTIVE DIMENSION TWO

BY

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ABSTRACT. We prove the ideals associated with the construction of generic complexes are prime and arithmetically perfect. This is used to construct the generic resolution for modules of projective dimension two.

Introduction. Let R be a commutative Noetherian ring with identity and I an ideal of R. If the grade of I (the length of the longest R-sequence in I) is equal to the projective dimension of I, I is said to be "perfect". The associated primes of a perfect ideal I all have the same grade as I. If R is Cohen-Macaulay, and I a perfect ideal, then R/I is again Cohen-Macaulay. The classical example of a perfect ideal is due to Macaulay: in a polynomial ring over a field, any ideal of height n which is generated by n elements is perfect.

If R is regular and R/I has no nontrivial idempotents, then I is perfect if and only if R/I is Cohen-Macaulay. In particular, if R is a polynomial ring over the integers or over a field and I is a homogeneous ideal this holds.

Several large classes of perfect ideals are known, especially among "determinantal" ideals. The main example is the $t \times t$ minors of a generic $r \times s$ matrix as was established by Eagon and Hochster [15] and Musili [39]. For other work, see [1], [5], [9], [13], [14], [16], [19], [21], [24], [26], [31], [34], [35], [36], and [46].

In the above papers four main methods have arisen to prove a given ideal I in a polynomial ring is perfect. The first of these is the actual computation of the projective resolution of I as was done in [9], [16], [19], and [37], and of which the Kozsul complex may be seen as the queen-mother. In practice this method is the most difficult; even the resolution of the maximal minors of a generic $r \times s$ matrix remains unknown over the integers; in characteristic 0 this has been solved by Lascoux [37].

The second method was realized by Hochster and Roberts in [24] where they show:

THEOREM 1.1A [24]. *Let G be a linearly reductive affine linear algebraic group over a field K of arbitrary characteristic acting K-rationally on a regular Noetherian K-algebra S. Then the ring of invariants $R = S^G$ is Cohen-Macaulay.*

This result was later generalized by Boutot who proved

THEOREM 1.2A (BOUTOT [5]). *Let R and S be essentially of finite type over a field of characteristic* 0 *and suppose*
(1) *R is pure in S,*
(2) *S has rational singularities.*
Then R has rational singularities.

These results are extremely powerful and subsume many of the previous results. However this method does have two drawbacks; firstly it only works in characteristic zero for most examples and secondly the class of perfect ideals I such that R/I can be represented as the invariants of a linearly reductive group is probably only a small subclass of all perfect ideals.

The third method has been recently developed by DeConcini, Eisenbud and Procesi ([12]–[14]) as an offshoot of invariant theory techniques and has the added interest of working in all characteristics. This approach consists basically of finding a new **Z**-basis for the ring $\mathbf{Z}[X_1, \ldots, X_n]$ which has a "straightening law" and then representing the ideal I as a free **Z**-module over this basis. Again, this method has the drawback of not working in the generality one often needs.

The fourth method (although in fact this was the second method chronologically) was developed by Hochster and Eagon in [15], and has since been used extensively. (See [21], [26], [34], and [35].) This technique, which Eagon and Hochster call principal radical systems, consists of finding a class of ideals, closed under suitable cuttings with hypersurfaces, whose maximal elements (under inclusion) are perfect, and along which one can descend the property of perfection. In practice, this method is the most widely applicable and handles the arithmetic case. In §2 we give a detailed description of principal radical systems, suitably modified to fit our own specific problems.

The main purpose of this paper is to find two new classes of perfect ideals, the first of which answers queries of Kempf and Eisenbud, the second of which completes the discussion of Hochster in [22]. We now describe these questions in detail.

Let $V_0, \ldots, V_n$ be a sequence of vector spaces. (See Kempf [33] for the following discussion.) Let V be the direct sum of $\mathrm{Hom}(V_n, V_{n-1}), \ldots, \mathrm{Hom}(V_1, V_0)$. A point in V is denoted $(a_n, \ldots, a_1)$ where a_i is an element of $\mathrm{Hom}(V_i, V_{i-1})$. A point a in V represents a complex if $a_{i+1}a_i = 0$ for $0 < i < n$. The rank of a is the sequence of integers (rank $a_1, \ldots,$ rank a_n), where rank b is the dimension of the image of the homomorphism b. If $(m_n, \ldots, m_1)$ is the rank of a complex, then $m_n \leqslant \dim V_n$, $m_1 \leqslant \dim V_1$ and $m_i + m_{i+1} \leqslant \dim V_i$ for $0 < i < n$. Conversely, any such sequence is the rank of a complex. Let M be the set of such sequences.

If $m \in M$, define the variety of Buchsbaum and Eisenbud, B-E(m); to be the variety of complexes a such that rank $a_i \leqslant m_i$ for $1 \leqslant i \leqslant n$.

In [32], Kempf showed:

THEOREM 1.3A. *In characteristic* 0, B-E(m) *is a normal Cohen-Macaulay variety and has rational singularities.*

Kempf states this result should hold in any characteristic and in addition, the ideal of functions vanishing on B-E(m) should be generated by the quadric relations corresponding to the complex condition and the determinants corresponding to the rank conditions. We prove both of these results in §6.

THEOREM 6.0. *Let* $(a_n, \ldots, a_0)$ *be a sequence of nonnegative integers and* $(r_n, \ldots, r_1) = r$ *a sequence of positive integers such that* $r_n \leqslant a_n$, $r_1 \leqslant a_1$ *and* $r_i + r_{i-1} \leqslant a_{i-1}$. *Let* X_i *be a generic* $a_i \times a_{i-1}$ *matrix and suppose the entries of* $X_n, \ldots, X_1$ *are algebraically independent. By* $I_t(X_i)$ *denote the ideal generated by the* $t + 1 \times t + 1$ *minors of* X_i. *Let* I *be the ideal in* $R = \mathbf{Z}[X_n, \ldots, X_1]$ *generated by the entries of* $X_i X_{i-1}$ *for* $0 < i \leqslant n$ *and by all the* $I_{r_i}(X_i)$. *Then* R/I *is a Cohen-Macaulay domain and is the homogeneous coordinate ring of* B-E(r).

In fact, to show this result we need to show considerably more. The full class of perfect ideals is given by the next result.

THEOREM 6.1. *Let the notation be as in Theorem 6.0. Let* $s = (s_0, \ldots, s_m)$ *be a strictly increasing sequence of nonnegative integers such that* $s_0 = 0$, $s_m = a_0$ *and* $m \leqslant r_1$. *Let* J *be the ideal in* $R = \mathbf{Z}[X_n, \ldots, X_1]$ *generated by* I, *all* $t + 1 \times t + 1$ *minors of the first* s_t *columns of* X, *and the first* s_h *entries along the top row of* X_1. *Then* R/J *is a Cohen-Macaulay domain. We call such a* J *an ideal of* B-E *type.*

This result clearly mimics the work of Eagon and Hochster in [15]. A special case of Theorem 6.1 was shown in [15]; essentially they show:

PROPOSITION 1.1A [15]. *Let* X *be a generic* $r \times s$ *matrix with* $r < s$ *and* Y *a generic* $s \times 1$ *matrix such that the entries of* X *and* Y *together are algebraically independent. Then if* I *is the ideal in* $\mathbf{Z}[X, Y]$ *generated by the entries of* XY, I *is a perfect prime ideal of grade* r.

Both of the above theorems can be extended to an arbitrary Noetherian ring using the well-known technique of generic perfection developed by Northcott, Eagon and Hochster in [17], [40], [41] and [20].

We make this extension in detail in §6.

We do not prove the normality of the rings R/J, where J is of B-E type in full generality although a detailed computation would provide this. Instead, we only compute the normality in the case we discuss in the second half of this paper; resolutions of length two.

The second question we address deals with the generic family of rings constructed by Hochster [22] for resolutions of length 2. We describe his result briefly.

We will say a free resolution **F**: $0 \to F_2 \to F_1 \to F_0$ is of type $(b_2, b_1, b_0) = b$ if rank $F_i = b_i$. A pair $(S, \mathbf{F})$ is said to be universal of type b if S is a Noetherian ring, **F** a free S-resolution and if for any commutative Noetherian ring R and any resolution **G** over R of type b with basis chosen, there exists a unique map $f: S \to R$ such that $\mathbf{G} = \mathbf{F} \otimes_S R$.

In [22], Hochster constructed such universal generic pairs $(S, \mathbf{F})$ for resolutions of length 2. We briefly sketch his proof: let $b = (b_2, b_1, b_0)$ and let X be a $b_2 \times b_1$

matrix, Y a $b_1 \times b_0$ matrix such that the entries of X and Y are algebraically independent. Suppose $b_2 < b_1 \leq b_2 + b_0$ and set $r = (b_2, b_1 - b_2)$. Let I be the Buchsbaum-Eisenbud ideal of type r in $\mathbf{Z}[X, Y]$ and set $R = \mathbf{Z}[X, Y]/I$. By Theorem 6.1, R is a Cohen-Macaulay normal ring and there is a complex over R,

$$0 \to F_2 \overset{\overline{X}}{\to} F_1 \overset{\overline{Y}}{\to} F_0 \tag{1}$$

where F_i is a free R-module of rank b_i and $\overline{X}$ and $\overline{Y}$ denote the images of the matrices X and Y in R. By construction, rank $\overline{X}$ = rank F_2 and rank $\overline{X}$ + rank $\overline{Y}$ = rank F_1.

We would like (1) to be exact. For this, there is the criteria of Buchsbaum and Eisenbud [7].

THEOREM 1.4A. *Let R be a commutative Noetherian ring and*

$$F: 0 \to F_n \overset{f_n}{\to} \cdots \overset{f_1}{\to} F_0$$

a complex of free R-modules. Then F is exact if and only if the following two conditions are satisfied:

(1) *rank f_i + rank f_{i-1} = rank F_i.*

(2) *If $I(f_i) = I_{\text{rank } f_i}(f_i)$ then grade $I(f_i) \geq i$.*

Hence for (1) to be exact we need grade $I(\overline{Y}) \geq 1$ and grade $I(\overline{X}) \geq 2$. As R is a domain the former is satisfied, but the latter condition fails for obvious reasons: the multipliers (as in Buchsbaum-Eisenbud [8]) are not in the ring R.

Specifically let σ be a b_2 subset of $\{1, \ldots, b_1\}$, and let $\bar{\sigma}$ be the complementary set. Then σ determines a $b_2 \times b_2$ minor of X by taking the minor determined by the columns of X corresponding to elements of σ. Denote this minor by f_σ. If τ is any $b_1 - b_2$ element subset of $\{1, \ldots, b_0\}$ then $\bar{\sigma}$ and τ naturally determine a $(b_1 - b_2) \times (b_1 - b_2)$ minor of Y by taking rows corresponding to $\bar{\sigma}$ and columns corresponding to τ. We denote this minor by $g_{\sigma,\tau}$.

Buchsbaum and Eisenbud have shown that if R is a commutative Noetherian ring and $0 \to F_2 \overset{X}{\to} F_1 \overset{Y}{\to} F_0$ is an exact sequence of free R-modules of type $b = (b_2, b_1, b_0)$ then the elements $g_{\sigma,\tau}/f_\sigma \in R$ and $f_\sigma g_{\sigma',\tau} = f_{\sigma'} g_{\sigma,\tau}$.

We return to our situation. Clearly the $g_{\sigma,\tau}/f_\sigma$ of X and Y are not in R and so $I_{b_2}(\overline{X}) \subseteq (g_{\sigma,\tau}: f_\sigma)$ but $f_\sigma \notin R g_{\sigma,\tau}$. This shows $I_{b_2}(\overline{X})$ has grade 1.

Thus, we let S = the integral closure of $R[g_{\sigma,\tau}/f_\sigma]$ where σ runs through all b_2-element subsets of $\{1, \ldots, b_1\}$ and τ runs through all $(b_1 - b_2)$-element subsets of $\{1, \ldots, b_0\}$. Hochster shows,

THEOREM 1.5A [22]. *The pair $(S, \mathbf{F})$,*

$$\mathbf{F}: 0 \to F_2 \overset{\overline{X}}{\to} F_1 \overset{\overline{Y}}{\to} F_0$$

is a universal generic pair of type $b = (b_2, b_1, b_0)$.

Hochster points out that our knowledge of modules of projective dimension two nonetheless remains incomplete; for if $S \neq R[g_{\sigma,\tau}/f_\sigma]$ then the integral equation for an element of S not in $R[g_{\sigma,\tau}/f_\sigma]$ must correspond to additional structure of any free resolution of length 2.

The question whether $R[g_{\sigma,\tau}/f_\sigma] = S$ is easily seen (see §7) to be one of whether or not $R[g_{\sigma,\tau}/f_\sigma]$ satisfies Serre's condition S_2. In particular, if $R[g_{\sigma,\tau}/f_\sigma]$ were Cohen-Macaulay, then it would indeed be the generic ring and there would be no additional structure.

The normality of a ring of this type follows from the analytic behavior of a suitable ideal. More precisely,

PROPOSITION 8.1. *Let R be a Noetherian normal domain and $I = (a_1, \ldots, a_n, b)$ an ideal such that $\mathrm{gr}_I(R)$ is reduced. Then $S = R[a_1/b, \ldots, a_n/b]$ is normal.*

If I is a height one prime and $I^{(n)} = I^n$ for all n, $\mathrm{gr}_I(R)$ is a domain and so $S = R[a_1/b, \ldots, a_n/b]$ is normal.

We apply Proposition 8.2 to the situation described above by letting $b = f_\sigma$ where $\sigma = (n - r + 1, \ldots, n)$ and the a_i are the $g_{\sigma,\tau}$, and R is as above. The question becomes one of deciding whether or not $(f_\sigma, g_{\sigma,\tau})^{(n)} = (f_\sigma, g_{\sigma,\tau})^n$.

The literature concerning symbolic powers is extensive; for example see [23], [11], [28], [29], [12], [1], [30], [44], or [45]. There are few general methods for determining if $p^{(n)} = p^n$ for all n given a prime p, but the method of weak d-sequences established in [28] handles most cases connected with determinantal phenomena.

Let H be a finite poset, and $\{x_\alpha\}$ a set of elements in a commutative ring R indexed by α in H. If I is an ideal of R, I is said to be an H-ideal if I is generated by some subset $\{x_\lambda | \lambda \in \Lambda\}$ of the $\{x_\alpha\}$ and Λ has the property that $\beta \in \Lambda$ and $\alpha < \beta$ imply $\alpha \in \Lambda$. We set $I_\alpha = \langle x_\beta | \beta < \alpha \rangle$, which is clearly an H-ideal. If J is an ideal, by J^* we denote the ideal generated by all $x_\beta \in J$. We set $K = \langle x_\alpha | \alpha \in H \rangle$.

DEFINITION. Let R, H, $\{x_\alpha\}$ be as above. We will say the $\{x_\alpha\}$ form a weak d-sequence if whenever I is an H-ideal containing I_α but not x_α then

(1) $(I : x_\alpha) \cap K = (I : x_\alpha)^*$ is an H-ideal.
(2) If $x_\beta \in (I : x_\alpha)$ then $x_\alpha x_\beta \in IK$.
(3) If $x_\alpha \notin (I : x_\alpha)$ then $(I : x_\alpha) = (I : x_\alpha^2)$.

Of course, if all H-ideals are self-radical, (3) will automatically hold. If $\{x_\alpha\}$ are a weak d-sequence, an ideal J is said to be related to $\{x_\alpha\}$ if J is of the form $(I : x_\alpha) + K$ where I is an H-ideal containing I_α but not x_α. The main result of [28] is:

THEOREM 1.5. *Let R be a commutative ring and $\{x_\alpha | \alpha \in H\}$ a weak d-sequence in R. Let I be the ideal generated by all the x_α. Then for all n, R/I^n has a filtration $R/I^n = M_0 \supseteq \ldots \supseteq M_d = (0)$ of R-modules such that M_k/M_{k+1} is isomorphic to R/J where J is a related ideal of the x_α.*

COROLLARY. *Suppose R is a commutative ring and $\{x_\alpha | \alpha \in H\}$ a weak d-sequence. Suppose the ideal I generated by all x_α is prime and $I_q^{(n)} = I_q^n$ for prime ideals q associated to any related ideal. Then $I^{(n)} = I^n$.*

We show the ideal of maximal minors of the last $b_1 - b_2$ rows of Y and the maximal minor of X determined by the last b_2 columns form a weak d-sequence. This is extremely easy from the principal radical system of §4 for the Buchsbaum-Eisenbud ideals. Not only is this verification easy, but all the related ideals are

prime ideals in the principal radical system and the various localizations are easy to compute.

Throughout this paper, if K is a ring and X a matrix with coefficients in some ring S containing K, $K[X]$ will mean the subring of S generated by the entries of X. For basic terminology and concepts such as grade or depth, we refer the reader to Matsumura [38]. All rings are commutative with one. Any result used from another paper will be denoted with an "A", i.e. Theorem 1.6A.

2. Principal radical systems. The concept of principal radical systems (PRS) was developed by Eagon and Hochster in [15]. The form has been modified both by Hochster [21] and others. For the purposes of this paper we use the following definition and lemma.

DEFINITION 2.1 (PRS). Let H be a finite poset and x_i a set of elements in a commutative ring R indexed by $i \in H$. Set $X = \langle x_i \rangle$. Let W be a finite collection of ideals in R satisfying the following properties:

If $I \in W$ and, for all $j < i$, $x_j \in I$ then $(I, x_i) \in W$, and there is a $J \in W$ such that $I \subseteq J \subseteq (I : x_i)$ and x_i is a nonzero divisor on $R/\text{rad}(J)$. (Here $\text{rad}(J)$ is the nilradical.) Further suppose that $\cap (I, x_i^n) = I$. (This automatically holds in the local or homogeneous case.) Then the collection W is said to be a principal radical system (PRS).

LEMMA 2.1. *Let W be a PRS and suppose, for all $I \in W$, $I + X$ is self-radical. Then*

(1) *All $I \in W$ are self-radical.*

(2) *If $I \in W$ and $x_j \in I$ for all $j < i$ then $(I : x_i) \in W$ and $(I : x_i) = (I : x_i^2)$.*

(3) *If there is an ideal Q which is not contained in any associated prime of $I + X$ for any $I \in W$, then Q is not contained in any associated prime of I for any $I \in W$.*

PROOF. We prove (1) first. Let $I \in W$ be a maximal element of W (under inclusion) which is not self-radical. If $X \subseteq I$, then $I = I + X$ is self-radical by assumption so we may suppose X is not contained in I. As H is finite there is a least $i \in H$ such that $x_i \notin I$. Then $(I, x_i) \in W$ and is larger than I so by induction is self-radical. Further let $J \in W$ be such that $I \subseteq J \subseteq (I : x_i)$ and x_i is not a zero-divisor on $R/\text{rad } J$. Since $\text{rad}(I) \subseteq \text{rad}(I, x_i) = (I, x_i)$ we see

$$\text{rad}(I) \subseteq I + x_i(\text{rad } I : x_i).$$

Case 1. Suppose $J \neq I$. Then since J is larger than I and in W, J is self-radical and x_i is a nonzero divisor on R/J. But then $(\text{rad } I : x_i) \subseteq (\text{rad } J : x_i) = \text{rad } J = J$ and so

$$\text{rad}(I) \subseteq I + x_i J \subseteq I \quad \text{as } J \subseteq (I : x_i).$$

Case 2. Suppose $J = I$. Then $(\text{rad}(I) : x_i) = \text{rad } I$ and $\text{rad}(I) \subseteq I + x_i \text{ rad}(I)$. But then $(\text{rad } I) \subseteq \cap (I, x_i^n) = I$. This proves (1).

For (2) we show that the ideal J must equal $(I : x_i)$ and so $(I : x_i) \in W$. Since J is self-radical, x_i is not a zero-divisor on R/J. But if $y \in (I : x_i)$ then $x_i y \in I \subseteq J$ and so $y \in J$. This establishes our claim, as x_i is not a zero-divisor on R/J.

Finally we prove (3). Let I be a maximal ideal (under inclusion) such that Q is contained in an associated prime P of I. If $X \subseteq I$ this contradicts the assumption so that there is a least i in H such that $x_i \notin I$.

Case 1. $I \neq (I: x_i)$. Then since $(I: x_i) = (I: x_i^2)$ by step 2, we obtain

$$I = (I, x_i) \cap (I: x_i).$$

Each of these ideals is self-radical and both the ideals on the right are strictly larger than I. Since no prime in either of their decompositions contains Q, the same holds for I.

Case 2. $I = (I: x_i)$. We may assume there is a $c \in R$ such that $c \notin I$ but $cQ \subseteq I$. Then $cQ \subseteq (I, x_i)$ shows $c \in (I, x_i)$ by the choice of I. Hence $c = i_1 + c_1 x_i$ and $c_1 x_i Q \subseteq I$. Since $(I: x_i) = I$, this shows $c_1 Q \subseteq I$ and so $I + (I: Q)x_i = (I: Q)$. However this implies

$$(I: Q) \subseteq \bigcap_{n=1}^{\infty} (I, x_i^n)$$

and so $(I: Q) = I$ which shows Q is not contained in P.

It is worth noting the hard step in this process. This consists firstly of finding a large enough class of ideals to contain all the necessary ones and secondly proving that x_i is not a zero-divisor on $R/\mathrm{rad}(J)$. This is usually done by finding a "generic" point for such J, that is a mapping of R/J into a domain D such that the kernel of the mapping has the same radical as J and such that x_i does not go to zero.

Principal radical systems are useful not only to prove the primeness of a given ideal but also are useful in proving the perfection of the prime ideals they contain. One can descend the property of perfection along the PRS using various easy lemmas from homological algebra as was done in [3] and [15]. We list the necessary lemmas here. (See [2].)

LEMMA 2.2A. *If I and J are perfect ideals of grade g in a Noetherian ring R, and $I + J$ has grade $g + 1$, then $I + J$ is perfect if and only if $I \cap J$ is perfect (necessarily of grade g).*

LEMMA 2.3A [21]. *Let $I_1, \ldots, I_k$ be perfect ideals of grade g in a Noetherian ring R and suppose for any i, $1 \leqslant i \leqslant k$, the sum of any i of these ideals has grade $(g - 1) + i$. Suppose also that the least family of ideals generated from $I_1, \ldots, I_k$ under the operations $+$, $\cap$ is a distributive lattice under $+$, $\cap$. Then the following two conditions are equivalent:*

(1) *The sum of every subset of the ideals $I_1, \ldots, I_k$ is perfect.*
(2) *The intersection of every subset of the ideals $I_1, \ldots, I_k$ is perfect.*

The usefulness of this comes from the following observation.

LEMMA 2.4. *Let W be a PRS and suppose*
(1) *If $I \in W$ then (I, X) is self-radical.*
(2) *If $I, J \in W$ both $I + J$ and $I \cap J \in W$.*
Then the ideals in W form a distributive lattice under $+$, $\cap$.

PROOF. It is enough to observe that if $I_1, \ldots, I_k, J \in W$ then the ideals $(I_1 \cap \ldots \cap I_k) + J$ and $(I_1 + J) \cap \ldots \cap (I_k + J)$ have the same radical and are both in W. By Lemma 2.1, these ideals are self-radical and hence are equal.

3. Buchsbaum-Eisenbud varieties. In this section we begin the proof of the perfection of the Buchsbaum-Eisenbud ideals by embedding them in a suitably large principal radical system. The argument will consist of three steps: first the definition of the principal radical system and the definition of the ideal J required (as in Definition 2.1), second the description of the generic points, and finally the calculation of the grade of the ideals in the principal radical system (by finding new generic points) and the application of Lemma 2.3A to descend the property of perfection.

We now fix positive integers $(a_n, \ldots, a_0) = a$ and nonnegative integers $r = (r_n, \ldots, r_1)$ such that $r_n \leqslant a_n$, $r_1 \leqslant a_1$ and $r_i + r_{i-1} \leqslant a_i$ for $1 < i < n$. Fix matrices $X_n, \ldots, X_1$ with algebraically independent entries over a field K (which we fix), X_i of size $a_i \times a_{i-1}$ for $1 \leqslant i \leqslant n$. If X is a matrix we write $(X|i)$ to denote the first (left-most) i columns of X. We will write $(i|X)$ to denote the matrix formed from the bottom i rows of X and throughout $I_t(X)$ will denote the ideal generated by all the $t + 1$ by $t + 1$ minors of X. The entries of X_1 we denote simply by x_{ij}; the entries of X_k for $k > 1$ we denote by $x_{ij}^{(k)}$.

For the purposes of the second problem we wish to consider in this paper two different principal radical systems will be considered simultaneously. We fix for the remainder of the discussion two ideals contained in $R = K[X_n, \ldots, X_1]$. Define

$$L_1 = \sum_{i=1}^{n} I_{r_i}(X_i) + \sum_{i=1}^{n} I_1(X_i X_{i-1}),$$

and define $L_2 = L_1 + R\Delta$ where Δ is the $a_n \times a_n$ minor of X_n given by the last a_n rows of X_n. (Notice if $r_n < a_n$ then $\Delta \in L_1$.)

We now consider additional nonnegative integers $s = (s_0, \ldots, s_m)$, $0 = s_0 < \ldots < s_m = a_0$ where $m \leqslant r_1$. We let T be the set of such sequences and if $s \in T$ and p is an integer $0 \leqslant p \leqslant a_0$ we define

$$I(a, r, s, p) = L_1 + \sum_{t=0}^{m} I_t(x_1|s_t) + R(x_{11}, \ldots, x_{1p})$$

and define

$$I^2(a, r, s, p) = L_2 + \sum_{t=0}^{m} I_t(x_1|s_t) + R(x_{11}, \ldots, x_{1p}).$$

Set $W = \{I(a, r, s, p)|s \in T \text{ and } 0 \leqslant p \leqslant a_0\}$ and $W^2 = \{I^2(a, r, s, p)|s \in T \text{ and } 0 \leqslant p \leqslant a_0\}$.

We will show both W and W^2 are principal radical systems. For the partially ordered set H belonging to the principal radical system we take $x_{11}, \ldots, x_{1s}$ ordered linearly by $x_{11} < \ldots < x_{1s}$.

To show W and W^2 are principal radical systems we first observe that all the ideals are homogeneous and hence the condition $\cap(I, x_i^n) = I$ is immediate. If $I(a, r, s, p) \in W$ then $(I(a, r, s, p), x_{1p+1}) = I(a, r, s, p + 1) \in W$. Finally suppose

$I(a, r, s, p) \in W$ and $s_h \leqslant p < s_{h+1}$. We define an $s' \in T$, $s' = \{s'_0, \ldots, s'_m\}$ where $s = \{s_0, \ldots, s_m\}$ by

$$s'_j = s_j \quad \text{if } j \neq h,$$

$$s'_h = p.$$

Clearly $I(a, r, s, p) \subseteq I(a, r, s', p)$. We will show $I(a, r, s', p)$ satisfies the conditions necessary for the ideal J of Definition 2.1.

For the remainder of this discussion since $a = (a_n, \ldots, a_0)$ and $r = (r_n, \ldots, r_1)$ are fixed, we let $I(a, r, s, p) = I(s, p)$.

LEMMA 3.1. *Let the notation be as above. Then* $I(s', p) \subseteq (I(s, p) : x_{1p+1})$.

PROOF. From the definition of s', it is clear we need only show that if Δ is any $h + 1 \times h + 1$ minor of the first p rows of X_1, then $x_{1p+1}\Delta \in I(s, p)$. Δ is determined by $h + 1$ rows R and $h + 1$ columns C of X_1. If $\{1\} \subset R$, then $\Delta \in I(s, p)$. If we let $R' = \{1\} \cup R$ and $C' = \{p + 1\} \cup C$ then the sets R' and C' determine an $h + 2 \times h + 2$ minor W of the first $p + 1$ rows. Since $p < s_{h+1}$, $W \in I(s, p)$. If we expand W along the first row, $W = x_{1p+1}\Delta + W'$ where $W' \in (x_{11}, \ldots, x_{1p})$. Since this shows $W' \in I(s, p)$ we see $x_{1p+1}\Delta \in I(s, p)$ as required.

Consequently, to show W is a principal radical system it is enough to show the radicals of the various $I(s, p)$ do not have irreducible components containing x_{1p+1}. For this we construct generic points for ideals $I(s, p) \in W$ such that $p = s_h$ for some s_h.

4. Generic points for $I(s, p)$. In this section we use the notation established in [15]. If X, X' are r by s and r by s' matrices respectively, we let $X \# X'$ be the r by $s + s'$ matrix formed by juxtaposing X and X'. If Y is an r' by r matrix clearly

$$Y(X \# X') = (YX) \# (YX').$$

If X is a matrix and K a ring we let $K[X]$ represent the ring K with the entries of X adjoined.

As in §3 we fix positive integers $a = (a_n, \ldots, a_0)$ and nonnegative integers $r = (r_n, \ldots, r_1)$ such that $r_n \leqslant a_n$, $r_1 \leqslant a_1$ and $r_i + r_{i-1} \leqslant a_{i-1}$ for $1 < i < n$. We let $X_n, \ldots, X_1 = X$ be matrices whose entries are algebraically independent over a field K and such that X_i is $r_i \times r_{i-1}$. As in §3, we let $s \in T$ and $0 < p \leqslant a_0$ such that $p = s_h$ for some s_h in H.

Now let A_k be an r_k by r_k $(2 < k < n - 1)$ matrix with indeterminate coefficients; in the future we always assume the entries of all such matrices taken together are algebraically independent. Let A_n be an a_n by r_n matrix of indeterminates.

If $2 \leqslant k \leqslant n$ we let B_k be the a_k by a_{k-1} matrix with A_k in the top right-hand corner and zeroes elsewhere. As $r_k \leqslant a_k$ and $r_k \leqslant a_{k-1}$, this is possible.

Let E_k be an a_k by a_k generic matrix for $1 \leqslant k \leqslant n$.

Now set $j_k = s_k - s_{k-1}$ $(1 \leqslant k \leqslant m)$ where $s = (s_0, \ldots, s_m)$. For each k in $\langle 1, m - 1 \rangle$ (by which we mean $\{1, \ldots, m - 1\}$) let U_k be a $k + 1$ by k matrix of indeterminates and let U_m be a $a_1 - r$ by m matrix of indeterminates: For k in

$\langle 1, m \rangle$ let V_k be a k by j_k matrix of indeterminates. Let W' be the $a_1 - r_2$ by s matrix $(U_m \ldots U_1 V_1)\# \ldots \#(U_m \ldots U_k V_k)\# \ldots \#(U_m V_m)$. Let W be the $a_1 \times a_0$ matrix with W' in its top $a_1 - r$ rows and zeroes elsewhere. Let Y be the first row of U_m and let $P = I_0(Y U_{m-1} \ldots U_n)$. We note the following proposition found in [15].

PROPOSITION 4.1A. *Let K be a Noetherian domain. Let $r = t_{m+1} > t_m > \ldots > t_1$ be a strictly decreasing sequence of positive integers. Let U_i be a t_{i+1} by t_i matrix of indeterminates over K, $1 \leqslant i \leqslant m$. Let Y be a 1 by r matrix of indeterminates. Then the entries of the 1 by t_1 product matrix $Y U_m \ldots U_1$ form an R-sequence in the polynomial ring generated over K by the indeterminates in Y and the U_i's, and the ideal generated by the entries is prime.*

Let L be the ring, $K[E_1, \ldots, E_n, (\det E_1)^{-1}, \ldots, (\det E_n)^{-1}]$ and define
$$D(s, p) = L[B_2, \ldots, B_n, U_m, \ldots, U_1, V_m, \ldots, V_1]/P.$$
By the above proposition $D(s, p)$ is a domain.

We let $R = K[X_n, \ldots, X_1]$ and define a map $f(s, p): R \to D(s, p)$ by sending the entries of X_n to the entries of $B_n E_n^{-1}$, the entries of X_j for $1 < j < n$ to the entries of $E_{j+1} B_j E_j^{-1}$ and the entries of X_1 to the entries of $E_1 W$.

PROPOSITION 4.1. *The map $f(s, p)$ defined above is a generic point for $I(s, p)$; that is $\ker(f(s, p))$ is a prime ideal and $\mathrm{rad}(I(s, p)) = \ker(f(s, p))$. Further x_{1p+1} is not a zero-divisor on $R/\ker(f(s, p))$.*

PROOF. The last statement follows from the first since $f(s, p)(x_{1p+1}) \neq 0$.

Let us first show that $I(s, p) \subseteq \ker(f(s, p))$. First, we notice that $B_{j+1} B_j = 0$ for $1 \leqslant j \leqslant n - 1$ since $r_i + r_{i-1} \leqslant a_{i-1}$. Hence
$$\left(E_{j+2} B_{j+1} E_{j+1}^{-1}\right)\left(E_{j+1} B_j E_j^{-1}\right) = 0 \quad \text{(where } E_{n+1} = E_0 = \text{the identity)}$$
and so $f(s, p)(X_{j+1} X_j) = 0$.

Next, clearly $I_{r_j}(B_j) = 0$ and so $I_{r_j}(E_{j+1} B_j E_j^{-1}) = 0$ which shows
$$I_{r_j}(X_j) \subseteq \ker f(s, p).$$

Finally, as $I_t(W|j) = I_t(W'|j)$ we only need to see that $I_t(W|s_t) = 0$.

However, $I_t(W|s_t)$ is just $(U_m \ldots U_1 V_1)\# \ldots \#(U_m \ldots U_t V_t) = U^* V^*$ where $U^* = (U_m \ldots U_t)$ is $a_1 - r$ by t and $V^* = (U_{t-1} \ldots U_1 V_1)\# \ldots \# V_t$ is t by $\Sigma_{i=0}^t (s_i - s_{i-1}) = s_t$. Hence $I_t(W|s_t) = I_t(U^* V^*)$ and
$$I_t(U^* V^*) \subseteq I_t(U^*) I_t(V^*) = 0.$$

It remains to show $\ker(f(s, p)) \subseteq \mathrm{rad}(I(s, p))$. Let "$\bar{}$" denote the images of elements of R modulo some prime $P \supset \mathrm{rad}(I(s, p))$. Let $D = K[X_n, \ldots, X_1]/P$ and let L be the fraction field of D. It will suffice to show there exist matrices B_j, E_j, U_j, V_j as above with coefficients in L such that each $\bar{X}_j$ can be factorized into the corresponding matrices.

LEMMA 4.1. *Suppose X and Y are matrices with coefficients in a field of sizes $r \times n$ and $n \times s$ respectively and $XY = 0$. Suppose rank $X \leqslant a$ and rank $Y \leqslant b$. Then there is an elementary matrix E such that XE^{-1} has zero entries in the first $r - a$ columns and such that $E^{-1} Y$ is zero in the last $n - b$ rows.*

PROOF. This is trivial using elementary column operations (in X) and elementary row operations (on Y).

We return to the proof of Proposition 4.1. First we apply Lemma 4.1 with $X = X_n$, $Y = X_{n-1}$, $a = r_n$ and $b = r_{n-1}$. Over L, there exists an elementary matrix E_n' such that $\overline{X}_n E_n'$ has zero entries in the first $a_n - r_n$ columns, and $(E_n')^{-1}Y$ is zero in the last $a_{n-1} - r_{n-1}$ rows. Thus there is an a_n by r_n matrix A_n' such that $\overline{X}_n = (0 \,\#\, A_n')(E_n')^{-1}$ where 0 is an a_n by $a_n - r_n$ matrix with zero entries.

One may choose E in such a way that the row space of $(t|X_{n-1}E^{-1})$ is the same as that of $(t|X)$.

We now apply Lemma 4.1 to the matrix $(E_n')^{-1}\overline{X}_{n-1}$ and $\overline{X}_{n-2}$. Since $I(s, p) \subseteq P$, rank $\overline{X}_{n-1} \leqslant r_{n-1}$, and rank $\overline{X}_{n-2} \leqslant r_{n-2}$ so there is an elementary matrix E_{n-1}' over L such that $B_{n-1}' = (E_n')^{-1}\overline{X}_{n-1}E_{n-1}'$ is zero in the first $a_{n-2} - r_{n-1}$ columns, and such that the row spaces of B_{n-1}' are the same as those of $(E_n')^{-1}\overline{X}_{n-1}$. In particular, the last $a_{n-1} - r_{n-1}$ rows of B_{n-1}' are zero. But then there is an r_{n-1} by r_{n-1} matrix A_{n-1}' such that B_{n-1}' is the a_{n-1} by a_{n-2} matrix with zeroes everywhere but in the upper right-hand corner, where A_{n-1}' is placed. Hence

$$\overline{X}_{n-1} = E_n'B_{n-1}'(E_{n-1}')^{-1}$$

has the necessary factorization. It is clear we may inductively choose the E_j' and B_j' with the proper form so that $\overline{X}_j = E_{j+1}'B_j'(E_j')^{-1}$ for $2 < j \leqslant n - 1$ and such that $E_1'\overline{X}_1$ is zero in the last $a_1 - r_1$ rows. Since E_1' does not change the column ranks of $\overline{X}_1$, the matrix $Z = E_1'\overline{X}_1$ is such that $I_t(Z|s_t) = 0$ since $I_t(\overline{X}_1|s_t) = 0$. Further if $Z = (z_{ij})$ we see $z_{11} = \ldots = z_{1p} = 0$. But now the necessary factorization of $E_1'\overline{X}_1$ follows from Proposition 36 of [15], which we state below.

PROPOSITION 4.2A [15]. *Let K be a Noetherian domain, X an r by s matrix of indeterminates over K and $R = U[X]$. Let $s = (s_0, \ldots, s_m)$ where $s_0 = 0$ and $s_m = s$. For $1 \leqslant k \leqslant m - 1$ let U_k be a $(k + 1)$ by k matrix of indeterminates over K and let U_m be an $r \times (m - 1)$ matrix of indeterminates. For each k, $1 < k \leqslant m$, let V_k be a k by $(s_k - s_{k-1})$ matrix of indeterminates and suppose all the entries of the U's and V's taken together are algebraically independent over K. Let W be the r by s matrix*

$$(U_m \ldots U_1 V_1)\# \ldots \#(U_m \ldots U_k V_k)\# \ldots \#(U_m V_m).$$

Let P be the ideal given by $I_0(YU_{m-1} \ldots U_k)$ where Y is the first row of U_m. Then the K-homomorphism ϕ mapping $K[X]$ onto $K[W]/P$ by taking the entries of X to the corresponding entries of W is a generic point for the ideal $J_{s,p}$ which is equal to $\sum_{t=1}^{m} I_t(x|s_t) + (x_{11}, \ldots, x_{1p})$.

COROLLARY 4.1. *W is a principal radical system.*

PROOF. By Lemma 3.1 the ideal $I(s', p)$ satisfies

$$I(s, p) \subseteq I(s', p) \subseteq \big(I(s, p): x_{1p+1}\big)$$

and $p = s_h'$ for some s_h' in s. By Proposition 4.1, $\ker(f(s', p))$ is a generic point for $I(s', p)$ and x_{1p+1} is not in $\ker(f(s', p))$. All the conditions of Definition 2.1 are now satisfied.

It is easy to modify the matrix E_n so that we obtain generic points for $I^2(s, p)$. Thus,

COROLLARY 4.3. W^2 is a principal radical system and if s and p are such that $p = s_h$ for some s_h in H, then $\mathrm{rad}(I^2(s, p))$ is prime.

THEOREM 4.1. Let $(a_n, \ldots, a_0) = a$ be any set of positive integers and $(r_n, \ldots, r_1) = r$ any set of nonnegative integers such that $r_n \le a_n$, $r_1 \le a_0$ and $r_i + r_{i-1} \le a_{i-1}$. Further let $0 = s_0 < \ldots < s_m = a_0$ be integers, $s = (s_0, \ldots, s_m)$ such that $m \le r_1$ and let p be an integer, $0 \le p \le a_0$. Let X_i be a_i by a_{i-1} matrices, $1 \le i \le n$, and let K be any field. Let $R = K[X_n, \ldots, X_1]$ and let $I(a, r, s, p)$ be the ideal in R defined by

$$I(a, r, s, p) = \sum_{i=1}^{n} I_{r_i}(X_i) + (\text{the entries of } X_i X_{i-1})$$

$$+ \sum_{t=0}^{m} I_t(X_1 | s_t) + (x_{11}, \ldots, x_{1p}).$$

Then $I(a, r, s, p)$ is self-radical and if $p = s_h$ for some $0 < h \le m$, $I(a, r, s, p)$ is prime.

PROOF. We induct on $\sum_{i=0}^{n} a_i = \tilde{a}$. By Proposition 3.1, for fixed a and fixed r, the ideals $I(a, r, s, p)$ form a principal radical system and so by Lemma 2.1 to show they are self-radical it is enough to show that, for each $I(a, r, s, p)$, $(I(a, r, s, p), x_{11}, \ldots, x_{1a_0})$ is self-radical.

However the ideal above is of the form $I(a', r', s', 0) + (x_{11}, \ldots, x_{1a_0})$ where $(a_n, \ldots, a_2, a_1', a_0) = a'$ and $r' = (r_n, \ldots, r_2, r_1')$ where $a_1' = a_1 - 1$ and $r_1' = r_1$ unless $r_1 = a_1$ in which case $r_1' = a_1'$ and where s' is suitably adjusted. As $\tilde{a}' < \tilde{a}$ we may conclude $I(a', r', s', 0)$ is self-radical and as $x_{11}, \ldots, x_{1a_0}$ are algebraically independent modulo this ideal we see $(I(a, r, s, p), x_{11}, \ldots, x_{1a_0})$ is self-radical. Since these ideals are self-radical and $\mathrm{rad}(I(a, r, s, p))$ is prime if $p = s_h$ for some h, the second statement follows immediately.

By the same method we see $I^2(a, r, s, p)$ is prime if $p = s_h$ for some h, $0 < h < m$.

5. Calculation of dimension. In this section we use the fact that the $I(a, r, s, s_h)$ are prime to find a new generic point for them from which one can calculate their dimension.

For this, let A_k be an $a_k - r_{k+1}$ by r_k matrix of indeterminates and B_k be the a_k by a_{k-1} matrix with A_k in the top right-hand corner and zeroes elsewhere. Let E_k' be an r_k by $a_{k-1} - r_k$ matrix of indeterminates and let E_k be the a_{k-1} by a_{k-1} matrix with E_k' in the lower left-hand corner, ones down the diagonal and zeroes elsewhere. Clearly $\det(E_k) = 1$. Let U be an $(a_1 - r_2)$ by m matrix whose first h rows are zero and let U_k be the kth column of U. Let V_k be a k by $(s_k - s_{k-1} - 1)$ matrix of indeterminates and let

$$W' = U_1 \# (U|_1) V_1 \# U_2 \# (U|_2) V_2 \ldots (U|_m) V_m.$$

Then let W be the a_1 by a_0 matrix with W' in the upper $a_1 - r_2$ rows and zero elsewhere.

PROPOSITION 5.1. *Let K be a field, and let $I(a, r, s, s_h)$, W, E_k, B_k be as above. Then the K-homomorphism $f: K[X_n, \ldots, X_1] \to K[W, E_k, B_k]$ which takes*

$$X_n \to B_n E_n^{-1},$$

$$X_j \to E_{j+1} B_j E_j^{-1} \quad for \ 1 < j \leqslant n,$$

$$X_1 \to E_2 W$$

is a generic point for $I(a, r, s, s_h)$; hence $\ker f = I(a, r, s, s_h)$. Moreover the fraction fields $K(B_n E_n^{-1}, E_j B_j E_{j-1}^{-1}, E_1 W)$ and $K(A_k, E_k, U, V_k)$ are equal so that

$$\dim I(a, r, s, s_h) = \sum_{k=2}^{n} (a_k - r_{k+1}) r_k + \sum_{k=2}^{n} r_k (a_{k-1} - r_k)$$

$$+ (a_1 + a_0 - r_2) m - \left(h + \tfrac{1}{2} m(m+1) + s_0 + \ldots + s_{m-1} \right)$$

where we set $r_{n+1} = 0$.

PROOF. By the construction of the B_j and W it is clear that $I(a, r, s, s_h) \subseteq \ker(f)$. To show the opposite inclusion it is enough to show the images of the X_i in the fraction field of the domain $D = K[X_n, \ldots, X_1]/I(a, r, s, s_h)$ can be factored into the same form as the map f takes. We do this as in the proof of Proposition 4.1, starting with X_n. It is easy to see by a specialization that $I_{r_n - 1}(\overline{X}_n) \neq 0$ in D where "$^-$" denotes the image in D. Hence the last r_n columns of $\overline{X}_n$ are linearly independent over $L = $ the fraction field of D. Since $I_{r_n}(\overline{X}_n) = 0$, every other column of $\overline{X}_n$ is a linear combination of the last r_n columns. Therefore there is a matrix E_n' over L of the form of E_n such that $\overline{X}_n E_n' = B_n'$ is a matrix with zeroes everywhere but the last r_n columns. Hence $\overline{X}_n = B_n'(E_n')^{-1}$ as required.

Consider the matrix $(E_n')^{-1}\overline{X}_{n-1}$. Since $(\overline{X}_n E_n')(E_n'^{-1}\overline{X}_{n-1}) = 0$ and $\overline{X}_n E_n' = B_n'$ is a matrix whose last r_n columns are linearly independent, the last r_n rows of $\overline{X}_{n-1}' = (E_n')^{-1}\overline{X}_{n-1}$ must be zero. As above, since $I_{r_{n-1}}(\overline{X}_{n-1}') = I_{r_{n-1}}(\overline{X}_{n-1}) = 0$, and since $(E_n')^{-1}$ does not affect the column space of $\overline{X}_{n-1}$, the last r_{n-1} columns of $\overline{X}_{n-1}$ (and hence $\overline{X}_{n-1}'$) are linearly independent and every column of $\overline{X}_{n-1}$ (and hence of $\overline{X}_{n-1}'$) is a linear combination of the last r_{n-1} columns. Therefore there exists a matrix E_{n-1}' of the form above such that $\overline{X}_{n-1}' E_{n-1}' = B_{n-1}'$ is a matrix of the form of B_{n-1}. Thus

$$(E_n')^{-1}\overline{X}_{n-1}E_{n-1}' = B_{n-1}' \quad and \quad \overline{X}_{n-1} = E_n' B_{n-1}' (E_{n-1}')^{-1}.$$

By an easy induction we see we may choose E_j', B_j' of the form required such that $\overline{X}_j = E_{j+1}' B_j' (E_j')^{-1}$ for $1 < j < n$ and such that $(E_1')^{-1}\overline{X}_1$ is a matrix $\overline{X}_1'$ whose last r_2 rows are zero, and whose entries are algebraically independent; we now apply Proposition 33 of [15].

PROPOSITION 5.1A [15]. *Let K be a field s, $p = s_h$ as above, and W', U, $V_1, \ldots, V_m$ as above. Then the K-homomorphism $g: K[\overline{X}_1'] \to K[W']$ is a generic point for the ideal*

$$I = \sum I_t(\overline{X}_1' | s_t) + (x_{11}, \ldots, x_{1p}).$$

Moreover the fraction fields $K(W')$ and $K(U, V_1, \ldots, V_m)$ are equal so that

$$\dim I = (a_1 + a_0 - r_2) m - \left(h + \tfrac{1}{2} m(m+1) + s_0 + \ldots + s_{m-1} \right).$$

This shows we may factor $\overline{X}'_{n-1}$ in the required form. To finish the proof it is enough to observe that $K(B_n(E_n)^{-1}, (E_n)(B_{n-1}E_{n-1}^{-1})) = K(B_n, E_n, B_{n-1}E_{n-1}^{-1})$ since in $K(B_n(E_n)^{-1}, (E_n)(B_{n-1}E_{n-1}^{-1}))$ we have $K(B_n, B_{n-1})$, and so may invert the minors of B_n and the minors of B_{n-1} and solve for the entries of E_n. Similarly

$$K\left(B_j E_j^{-1}, E_j\left(B_{j-1}E_{j-1}^{-1}\right)\right) = K\left(B_j, E_j, \left(B_{j-1}E_{j-1}^{-1}\right)\right),$$

and so these observations together with Proposition 5.1A [15] show the second claim of Proposition 5.1. The dimension count is then trivial, using Proposition 5.1A [15].

COROLLARY 5.1. *Suppose* $(a_n, \ldots, a_0)$ *is a sequence of nonnegative integers with* $a_n \leqslant a_{n-1}$ *and* $(r_n, \ldots, r_1)$ *and such that* $r_n = a_n$, $r_j + r_{j-1} = a_{j-1}$ *for* $1 < j < n$ *and* $r_1 \leqslant a_0$. *Let* $s_0 = 0$, $s_1 = 1, \ldots, s_{r_1-1} = r_1 - 1$, $s_{r_1} = a_0$, *and* $p = 0$. (*So that these conditions give a complex with the appropriate rank conditions for exactness.*) *Then*

$$\dim I(a, r, s, 0) = \sum_{j=1}^{n} r_j a_{j-1}.$$

PROOF. By Proposition 5.1 this is immediate.

6. The perfection of B-E(a). In this section we prove the perfection of the ideals $I(a, r, s, s_h)$ and further reduce the problem over a general ring to the generic case.

To this purpose we fix a and r and let $s = \{s_0, \ldots, s_m\}$ and $p = s_h$ vary, under the constraints of §3.

PROPOSITION 6.1. *Suppose* $s_h < p < s_{h+1}$. *Let* $s' = (s_0, \ldots, s_{h-1}, p, s_{h+1}, \ldots, s_m)$ *and let* $p' = s_{h+1}$. *Then*

$$I(a, r, s, p) = I(a, r, s', p) \cap I(a, r, s, p').$$

PROOF. Certainly $I(a, r, s, p)$ is contained in the intersection. If we can show $I(a, r, s', p)I(a, r, s, p') \subseteq I(a, r, s, p)$ then since $\mathrm{rad}(I(a, r, s', p)I(a, r, s, p')) = \mathrm{rad}(I(a, r, s', p) \cap I(a, r, s, p')) = I(a, r, s', p) \cap I(a, r, s, p')$ we will have proved the result. It suffices to show that for $p + 1 \leqslant k \leqslant s_{h+1}$, if M is any $h + 1$ by $h + 1$ minor of $(X_1|p)$, $x_{1k}M \in I(a, r, s, p)$. If the first row of M coincides with the first row of X_1 this is obvious. If not, let M^* be the unique $h + 2$ by $h + 2$ minor of X_1 which contains M and x_{1k}. $M^* \in I(a, r, s, p)$. The expansion by minors of M^* along the first row is a sum containing $x_{1k}M$ and all other terms are in $(x_{11}, \ldots, x_{1p})$. Hence $x_{1k}M \in I(a, r, s, p)$ as required.

THEOREM 6.1. *Let* K *be any field,* $R = K[X_n, \ldots, X_1]$. *Suppose* $m < a_1$ *and* $s_h < a_0$, *and let* s, s', p' *be defined as above for any* s, p. *Then both* $I(a, r, s, s_h)$ *and* $I(a, r, s, s_h + 1)$ *are perfect.*

PROOF. We use induction and Lemma 2.2A. Inducting on $\tilde{a} = \Sigma a_i$ allows us to assume the result for smaller matrices. Let $I(a, r, s, p)$ be maximal such that it is not perfect.

LEMMA 6.1A [21]. *Let* K *be a field and* I *a homogeneous ideal of the polynomial ring* $K[x_1, \ldots, x_n]$. *If* f *is a form of positive degree not a zero-divisor on* I, *then* I *is perfect if and only if* $I + fR$ *is perfect.*

If $p = s_h$ then by Theorem 4.1, $I(a, r, s, p)$ is prime and so x_{1_p+1} is not a zero-divisor on this ideal. But then the induction shows $I(a, r, s, p + 1)$ is perfect and from Lemma 6.1A we conclude $I(a, r, s, p)$ is perfect.

Thus we may suppose $p = s_h + 1$. By Proposition 6.1, $I(a, r, s, p) = I(a, r, s', p)$ $\cap$ $I(a, r, s, p')$ where $s' = \{s_0, \ldots, s_{h-1}, p, s_{h+1}, \ldots, s_m\}$ and $p' = s_{h+1}$. By induction both of these ideals are perfect. We may compute their dimension from Proposition 5.1:

$$
\dim I(a, r, s', p) = \sum_{k=2}^{n} r_k(a_k + a_{k-1} - r_k - r_{k+1}) + (a_1 + a_0 - r_2)m
$$
$$
- \left(h + \tfrac{1}{2}m(m + 1) + s_0 + \ldots + s_{h-1} + p + \ldots + s_{m-1}\right),
$$
$$
\dim I(a, r, s, p') = \sum_{k=2}^{n} r_k(a_k + a_{k-1} - r_k - r_{k+1}) + (a_1 + a_0 - r_2)m
$$
$$
- \left(h + 1 + \tfrac{1}{2}m(m + 1) + s_0 + \ldots + s_{h-1} + s_h + \ldots + s_{m-1}\right).
$$

Since $p = s_h + 1$, the dimensions (and hence the grades) of $I(a, r, s', p)$ and $I(a, r, s, p')$ are the same. Further, $I(a, r, s', p) + I(a, r, s, p') = I(a, r, s', p')$ and

$$
\dim I(a, r, s', p') = \sum_{k=2}^{n} r_k(a_k + a_{k-1} - r_k - r_{k+1}) + (a_1 + a_0 - r_2)m
$$
$$
- \left(h + 1 + \tfrac{1}{2}m(m + 1) + s_0 + \ldots + s_{h-1} + p + \ldots + s_h\right)
$$

and so

$$
\text{grade } I(a, r, s', p) = \text{grade } I(a, r, s, p') = \text{grade } I(a, r, s', p') - 1.
$$

Since all these ideals are perfect Lemma 2.2A shows

$$
I(a, r, s, p) = I(a, r, s', p) \cap I(a, r, s, p')
$$

is also perfect which completes the proof of Theorem 6.1.

By the same token we note the ideals $I^2(a, r, s, s_h)$ are perfect, as either $I^2(a, r, s, s_h) = I(a, r, s, s_h)$ or $I^2(a, r, s, s_h)$ is given by a hypersurface on $R/I(a, r, s, s_h)$.

We now apply the notions of generically perfect [17] to deduce the general case of Theorem 6.1 from the generic case above.

LEMMA 6.2A [21]. *If I is a homogeneous ideal in $Z[x_1, \ldots, x_n]$ where Z is the integers and the x_i indeterminates, and $Z[x_1, \ldots, x_n]/I$ is Z-flat, then I is strongly generically perfect of grade g if and only if $K \otimes_Z I$ is perfect of grade g in $K[x_1, \ldots, x_n]$ for every field K.*

LEMMA 6.3A [21]. *Let I be a proper ideal of $Z[x_1, \ldots, x_n]$ such that $Z[x_1, \ldots, x_n]/I$ is Z-flat, I is perfect and grade $I = g$. Suppose R is a Noetherian ring and $a_1, \ldots, a_n$ are elements of R. Let I' be the ideal of R given by the image of I under the homomorphism of $Z[x_1, \ldots, x_n] \to R$ sending x_i to a_i. Then grade $I' \leqslant g$ and if grade $I' = g$ then I' is a perfect ideal.*

THEOREM 6.2. *Let R be a commutative Noetherian ring and $M_n, \ldots, M_1$ matrices over R with M_i of size $a_i \times a_{i-1}$. Let $r = (r_k, \ldots, r_1)$, $s = (s_0, \ldots, s_m)$ and $p = s_h$ be given with $m \leqslant r_1$. Let $I(a, r, s, p)(M) = I$ be the ideal of R defined by the equations of M_i corresponding to those above. Then*

$$\text{grade } I \leqslant \sum_{k=2}^{n} r_k(a_k + a_{k-1} - r_k - r_{k+1})$$

$$+ (a_1 + a_0 - r_2)m - \left(h + \tfrac{1}{2}m(m+1) + s_0 + \ldots + s_{m-1}\right)$$

and if equality occurs, then I is perfect.

PROOF. By Lemmas 6.2A and 6.3A and the computation of Theorem 6.1 (independent of the field K) we need only to show $R = Z[X_n, \ldots, X_1]/I(a, r, s, s_h)$ is Z-flat. But this is if and only if R is torsion free, i.e. if and only if $Z \cap I(a, r, s, s_h) = (0)$. This, however, is clear.

7. Normality. Although one can show the normality of the varieties $R/I(a, r, s, s_h)$, to avoid unnecessary complication we only do this for resolutions of length 2 which satisfy the rank condition of exactness.

THEOREM 7.1. *Let X be an $r \times n$ matrix of indeterminates and Y an $n \times s$ matrix of indeterminates such that the entries of X and Y taken together are algebraically independent over Z, the integers. Suppose $r < n \leqslant r + s$. Let P be the ideal of $R = Z[X, Y]$ generated by the entries of XY and $I_{n-r}(Y)$. Then R/P is integrally closed.*

PROOF. First we show the theorem over an arbitrary field K. Let $S = K[X, Y]/I$ where I is the ideal generated by the entries of XY plus the ideal $I_{n-r}(Y)$. By Theorem 6.1, S is Cohen-Macaulay. Hence to show S is integrally closed it is enough to show the codimension of the singular locus of S is at least two. Let J be the ideal in $K[X, Y]$ generated by the entries of XY plus $I_{n-r-1}(Y)$. Let $\bar{J}$ be the image of J in S. We first show $\bar{J}$ is contained in the ideal defining the singular locus of S.

Let Δ be an $n - r$ by $n - r$ minor of Y. Without loss of generality, we may assume Δ is determined by the first $n - r$ rows and first $n - r$ columns, i.e. $\Delta = (1, \ldots, n - r | 1, \ldots, n - r)$. Suppose $i, j > n - r$. Then expanding the $n - r + 1$ by $n - r + 1$ minor $(1, \ldots, n - r, i | 1, \ldots, n - r, j)$ we see that in S_Δ, since this minor is zero, we may solve for $\bar{y}_{ij}$ in terms of the y_{km} with $k \leqslant n - r$ or $m \leqslant n - r$.

Now fix any row of X, say the mth. In S,

$$\overline{X}_{m1}\,\overline{Y}_{11} + \ldots + \overline{X}_{mn}\overline{Y}_{n1} \quad = \quad 0,$$

$$\vdots \tag{1}$$

$$\overline{X}_{m1}\overline{Y}_{1n-r} + \ldots + \overline{X}_{mn}\overline{Y}_{nn-r} \quad = \quad 0;$$

choose j, $1 \leqslant j \leqslant n - r$, and expand Δ along the jth row $\overline{Y}_{j1}, \ldots, \overline{Y}_{jn-r}$. Denote the $(n - r - 1)$ by $(n - r - 1)$ minor determined by

$$(1, \ldots, \hat{i}, \ldots, n - r \mid 1, \ldots, \hat{j}, \ldots, n - r)$$

by Δ_{ij}. Then $\Delta = \sum_{i=1}^{n-r} \overline{Y}_{ji} \Delta_{ji}$ while

$$\sum_{i=1}^{n-r} \overline{Y}_{ki} \Delta_{ji} = 0 \tag{2}$$

if $1 \leqslant k \leqslant n - r$ and $k \neq j$.

Multiplying the ith equation of (1) by Δ_{ji} and summing all the equations we obtain

$$\sum_{i=1}^{n-r} \Delta_{ji} \left(\sum_{k=1}^{n} \overline{X}_{mk} \overline{Y}_{ki} \right) = 0$$

and so

$$\sum_{k=1}^{n} \overline{X}_{mk} \left(\sum_{i=1}^{n-r} \overline{Y}_{ki} \Delta_{ji} \right) = 0. \tag{3}$$

Consider the inner sum of (3). If $1 \leqslant k \leqslant n - r$ and $k \neq j$ this sum is zero by (2). If $k = j$ the inner sum gives Δ. Hence

$$\Delta \overline{X}_{mj} \in \langle \overline{X}_{mk} \mid k > n - r \rangle.$$

Therefore, in S_Δ, $\overline{X}_{mj} \in \langle \overline{X}_{mk} \mid k > n - r \rangle$, as m was an arbitrary row of $\overline{X}$, i.e. $1 \leqslant m \leqslant r$. We see that in S_Δ, $\overline{X}_{mj}$ for $1 \leqslant m \leqslant r$, $1 \leqslant j \leqslant n - r$ can be solved for in terms of the $\overline{X}_{ik}$ with $k > n - r$.

Let Z_{ij} be new indeterminates, $1 \leqslant i \leqslant r$, $n - r + 1 \leqslant j \leqslant n$ and t_{ij} be new indeterminates, where we let $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant s$ but assume if $i, j > n - r$, then $t_{ij} = 0$, i.e. we do not add a new indeterminate for these i, j. Let $T = K[Z_{ij}, t_{ij}]$ and map $T \xrightarrow{f} S$ by Z_{ij} goes to X_{ij} and t_{ij} goes to Y_{ij}.

The discussion above shows if we let $\Delta' =$ minor of the t_{ij} given by $1 \leqslant i \leqslant n - r$, $1 \leqslant j \leqslant n - r$ (so that $f(\Delta') = \Delta$) then $T_{\Delta'} \xrightarrow{f'} S_\Delta$ is onto. As T and S are affine K-algebras, $\dim T = \dim T_{\Delta'}$ and $\dim S = \dim S_\Delta$. The $\dim T$ is easy to compute; there are r^2 Z_{ij} and $(n - r)r + (n - r)s$ t_{ij}. Hence

$$\dim T = rn + sn - rs.$$

We may calculate the $\dim S$ from Corollary 5.1 and we find $\dim S = rn + sn - rs$. Since f' is an onto map and $T_{\Delta'}$ is a domain of the same dimension as $f'(T_{\Delta'})$, we see f' is an isomorphism. Thus S_Δ is regular.

Let us compute the height of the ideal J generated by the entries of XY and those of $I_{n-r-1}(Y)$. This has a description (as in §4) $a_2 = r$, $a_1 = n$, $a_0 = s$, $r_2 = r$, $r_1 = n - r - 1$, $s_0 = 0, \ldots, s_{n-r-2} = n - r - 2$, $s_{n-r-1} = s$, $p = 0$. By Proposition 5.1, we compute

$$\text{ht}(J) = rs + r + s - n + 1.$$

Suppose $r + s > n$. Then $\text{ht}(J) \geqslant \text{ht}(I) + 2$ and so, in S, $\text{ht } JS \geqslant 2$. By our remarks above, this forces S to be integrally closed.

Hence suppose $r + s = n$. Then we claim that any r by r minor Δ of X is in the singular locus of S, i.e. S_Δ is regular. This easily follows from an argument similar to that above, observing $I_{n-r}(Y) = 0$ in this case. The ideal J generated by the entries of XY, $I_{r-1}(X)$, and $I_{n-r-1}(Y)$ has a description, $a_2 = r$, $a_1 = n$, $a_0 = s$, $r_2 = r - 1$, $r_1 = n - r - 1$, $s_0 = 0, \ldots, s_{n-r-2} = n - r - 2$, $s_{n-r-1} = s$, and $p = 0$. A calculation using Proposition 5.1 shows $\mathrm{ht}(J) = rs + 3$.

Hence $\mathrm{codim}(JS) = 3$ and again S is integrally closed.

To establish normality over Z, let Q be the rationals. Then $Q[X, Y]/I$ is integrally closed and so if $S = Z[X, Y]/I$ is not integrally closed there is an element f/c which is integral over S with $f \in S$ and $c \in Z$. Since S is faithfully flat over Z, $(c\colon f) \cap Z$ has grade at most one and is thus contained in a prime $p \in Z$. If we localize at (p), r/c is integral over $Z_{(p)} \otimes_Z S$ but not in the ring. Hence we may assume $V[X, Y]/I$ is not integrally closed where V is a discrete valuation ring. If t is a parameter for V we may assume there is an element f/t which is integral over $V[X, Y]/I$. If $(f/t)^n = \sum_{k=1}^{n-1} v_k(f/t)^k$ is such an equation of integral dependence, then t divides f^n. Thus it is enough to show tS is prime. But S/tS is a field L and $L[X, Y]/I$ is a domain by Theorem 4.1.

8. Generic resolutions of length 2. In this section we study the universal generic pair $(S, \mathbf{F})$ for resolutions of length two fixed Betti numbers. We refer the reader to the introduction, where the construction of this ring is carried through.

Throughout the rest of the section we fix generic matrices X and Y, X an r by n and Y an n by s matrix where we assume $s, r < n \leqslant r + s$. If σ is an r-element subset of $\langle 1, \ldots, n \rangle$, there is an associated r by r minor of X, f_σ, obtained by taking the columns corresponding to elements of σ. Of course σ also determines a complementary $n - r$ element subset which we denote by σ'. If τ is any $n - r$ element subset of Y, then σ and τ determine an $n - r$ by $n - r$ minor of Y, which we denote $g_{\sigma,\tau}$, by choosing $n - r$ rows of Y corresponding to elements of σ' and $n - r$ columns corresponding to τ. Buchsbaum and Eisenbud showed

THEOREM [8]. *Let R be a commutative Noetherian ring and*

$$0 \to R^r \xrightarrow{A} R^n \xrightarrow{B} R^s$$

an acyclic complex. Let f_σ and $g_{\sigma,\tau}$ be defined as above for the matrices A and B. Then there is a unique $c_\tau \in R$ such that for each σ in $T(r, n) = $ the set of r-element subsets of $\langle 1, \ldots, n \rangle$,

$$g_{\sigma,\tau} = \pm c_\tau f_\sigma.$$

The c_τ are called the multipliers and as one may assign an appropriate sign to the f_σ we may assume (as we will henceforth do) that $g_{\sigma,\tau} = c_\tau f_\sigma$.

In general

$$g_{\sigma,\tau} f_{\sigma'} = g_{\sigma',\tau} f_\sigma. \tag{1}$$

We now return to the generic case. Let I be the ideal in $Z[X, Y]$ defined by the entries of XY and $I_{n-r}(Y)$. I is clearly an ideal of Buchsbaum-Eisenbud type and Theorem 6.1 combined with Theorem 7.1 shows $R = Z[X, Y]/I$ is a Cohen-Macaulay normal domain. The equation (1) holds in R and so $I_{r-1}(\overline{X}) \subseteq (f_\sigma\colon g_{\sigma,\tau})$

for any fixed σ and τ, and hence grade $I_{r-1}(\overline{X}) \leqslant 1$. (Since R is integrally closed this shows $\mathrm{ht}(I_{r-1}(\overline{X})) = 1$.)

However by the Buchsbaum-Eisenbud criteria for exactness (see the introduction) if the complex $0 \to R^r \overset{\overline{X}}{\to} R^n \overset{\overline{Y}}{\to} R^s$ is to be exact, $I_{r-1}(\overline{X})$ must have grade at least 2.

Let us consider the ring $S = R[g_{\sigma,\tau}/f_\sigma]$ where we let σ be any element of $T(r, n)$ and τ any element of $T(n - r, s)$. (Of course from (1) we may assume σ is fixed since $g_{\sigma',\tau}/f_{\sigma'} = g_{\sigma,\tau}/f_\sigma$ for any σ and σ' in $T(r, n)$.) As in Hochster [22] it is easy to see that if $J = I_{r-1}(\overline{X})$ then $\mathrm{ht}(JS) \geqslant 2$, and so if S is either Cohen-Macaulay or integrally closed, $\mathrm{grade}(JS) \geqslant 2$ and the complex

$$\mathbf{F}: 0 \to S^r \overset{\overline{X}}{\to} S^n \overset{\overline{Y}}{\to} S^s$$

is exact and $(S, \mathbf{F})$ give the required universal pair. (See the introduction.)

In characteristic zero, Hochster [22] showed the universal ring S' is the ring of invariants of a linear algebraic group acting linearly on a polynomial ring. The group is not, however, linearly reductive so that Theorem 1.1A does not seem to apply. However Hochster (conversation) indicated that the difficulty can be circumvented by standard invariant theory. Consequently in characteristic 0, the universal generic ring S' is Cohen-Macaulay. Thus if we show S is integrally closed and hence equal to S', S *will be Cohen-Macaulay* (in characteristic zero).

We devote this section to a proof that S is normal, relating its normality to the normality of the blow-up of a prime p in R and this to the reducedness of the special divisor.

LEMMA 8.1 [4]. *Let R be an integrally closed Noetherian domain and I be an ideal such that $\mathrm{gr}_I(R)$ is reduced. Then $R(I) = R + It + I^2t^2 + \cdots$, the Rees ring of I, is normal.*

PROOF. This is proved in [4] but we take time to note an elementary proof. Let $R_{-1}(I) = R(I)[t^{-1}]$ which we shall refer to as the *extended Rees ring* (although this is often called the Rees ring). As is well known and easy to see, $R_{-1}(I)/(t^{-1}) \simeq \mathrm{gr}_I(R)$, and so (t^{-1}) is reduced in $R_{-1}(I)$. In particular if p is associated to (t^{-1}), $R_{-1}(I)_p$ is a discrete valuation ring. We also see $R_{-1}(I)_{t^{-1}} = R[t, t^{-1}]$ which is integrally closed. However,

$$R_{-1}(I) = R_{-1}(I)_{t^{-1}} \cap \bigcap_p R_{-1}(I)_p$$

where the second intersection runs over all primes associated to (t^{-1}). This shows $R_{-1}(I)$ is normal. Since $R[t]$ is normal and $R(I) = R_{-1}(I) \cap R[t]$ we see $R(I)$ is normal as required.

PROPOSITION 8.1. *Let R be a Noetherian normal domain and $I = (a_1, \ldots, a_n, b)$ an ideal such that $\mathrm{gr}_I(R)$ is reduced. Then $R[a_1/b, \ldots, a_n/b] = S$ is normal.*

PROOF. Let t be an indeterminate over R and consider $S[bt, (bt)^{-1}]$. It is clear this is the same ring as $R[a_1 t, \ldots, a_n t, bt, (bt)^{-1}]$ and this is just $R(I)_{bt}$. By Lemma

8.1, $R(I)$ is normal and hence so is $R(I)_{bt}$. However, bt is algebraically independent over S and so $S[bt, (bt)^{-1}] \simeq S[t, t^{-1}]$ and this being normal implies S is also.

We now return to our generic ring $R = Z[X, Y]/(XY = 0, I_{n-r}(Y))$ and $S = R[g_{\sigma,\tau}/f_\sigma]$. As we noted above we may assume $\sigma = \{n - r, \ldots, n\}$ and thus drop σ from our notation.

By Proposition 8.2, to show S is integrally closed it is enough to show the ideal $p = (f, g_\tau)_\tau$ satisfies the condition that its graded ring be reduced. The g_τ clearly run through the maximal minors of the first $n - r$ rows of Y.

We now place an order upon the τ in $T = T(n - r, s)$ by saying $(i_1, \ldots, i_{n-r}) \leqslant (j_1, \ldots, j_{n-r})$ if and only if

$$i_1 \leqslant j_1, \ldots, i_{n-r} \leqslant j_{n-r}.$$

T thus becomes a partially ordered set and we define *an ideal* $H \subset T$ to be any subset with the property that $\sigma \in H$ and $\sigma' \leqslant \sigma$ imply $\sigma' \in H$.

In a similar way if Y is an n by s matrix we may place partial order on *all* the minors of Y. Any k of k minor of Y is determined by k rows $i_1 < \ldots < i_k$ and k columns, $j_1 < \ldots < j_k$, and we denote this minor by $(i_1, \ldots, i_k | j_1, \ldots, j_k)$. We say

$$(i_1, \ldots, i_k | j_1, \ldots, j_k) \leqslant (i'_1, \ldots, i'_m | j'_1, \ldots, j'_m)$$

if and only if both $k \geqslant m$ and $i_1 \leqslant i'_1, \ldots, i_m \leqslant i'_m$, and $j_1 \leqslant j'_1, \ldots, j_k \leqslant j'_k$. (See [12], [13], or [14].) This order clearly agrees with that established above for the g_τ. We define an ideal in the same way. Let G be the partially ordered set defined by the above relation, and if $\sigma \in G$, let $\bar{\sigma}$ be the corresponding minor.

LEMMA 8.2 [13]. *Let Y be a generic n by s matrix over Z, the integers. Let G be as above and suppose $J \subset G$ is an ideal of G and $\sigma = (i_1, \ldots, i_k | j_1, \ldots, j_k)$ is an element which is minimal among elements of G not in J. If $J' \subset G$ is the set of all $\beta \in G$ such that $\beta \not\geqslant \sigma$, then $(\bar{J}: \bar{\sigma}) = \bar{J}'$.*

COROLLARY 8.1. *Let Y be as above and let $H \subset T$ be an ideal, where $T = T(n - r, s)$ is the subset of G determined by all elements of the form $(1, \ldots, n - r | i_1, \ldots, i_{n-r})$. Let $J = H \cup \{(i_1, \ldots, i_t | j_1, \ldots, j_t) | t \geqslant n - r + 1\}$. Finally, suppose $\sigma = (1, \ldots, n - r | i_1, \ldots, i_{n-r})$ is an element of T which is minimal above J. Then $(\bar{J}: \bar{\sigma}) = I(i_1, \ldots, i_{n-r})$ where $I(i_1, \ldots, i_{n-r})$ is the ideal defined by $\sum_{j=1}^{n-r} I_{j-1}(Y | i_j - 1) + I_{n-r}(Y)$.*

PROOF. From the partial order on G it is easy to see J is an ideal. By Lemma 8.2 $(\bar{J}: \bar{\sigma})$ is the set of all minors $\beta \in G$ such that $\beta \not\geqslant \sigma$. Let us first show $I_{t-1}(Y | i_t - 1)$ is in this set. Any t by t minor $\bar{\alpha}$ in $I_{t-1}(Y | i_t - 1)$ is given by $(j_1, \ldots, j_t | m_1, \ldots, m_t)$ where $m_t < i_t$. This shows $\alpha \not\geqslant \sigma$ and so $\bar{\alpha} \in (\bar{J}: \bar{\sigma})$. Since $I_{n-r}(Y)$ is contained in $\bar{J}$ we see $I(i_1, \ldots, i_{n-r}) \subseteq (\bar{J}: \bar{\sigma})$.

If $\alpha = (j_1, \ldots, j_t | m_1, \ldots, m_t) \in (\bar{J}: \bar{\sigma})$ then, by Lemma 8.2, $\alpha \not\geqslant \sigma$ and so since we may assume $t \leqslant n - r$, and either some $j_k < k$ or some $m_k < i_k$. The former cannot occur as $0 < j_1 < \ldots < j_k$. Thus the latter occurs and by expanding $\bar{\alpha}$ around its $t \times t$ minors we see that $\bar{\alpha} \in I_{t-1}(Y | i_k)$. This finishes the proof of Corollary 8.2.

Notice that the ideal $I(i_1, \ldots, i_{n-r})$ has a description (as above) by $s_0 = i_1 - 1$, $s_1 = i_2 - 1$, $s_{n-r-1} = i_{n-r} - 1$, $s_{n-r} = s$.

LEMMA 8.3. *Let X and Y be generic r by n and $n \times s$ matrices respectively, with $r < n \leqslant r + s$. Let $T = T(n - r, s)$ be the partially ordered set above and suppose $H \subset T$ is an ideal. Let $\sigma \in T$ be an element minimal over H. Let I be the ideal generated by the entries of XY, the r by r minor of X determined by its last r columns and $I_{n-r}(Y)$. Then $(I, \overline{H}: \overline{\sigma})$ is a prime ideal and $\overline{\sigma}$ is not a zero-divisor module in it.*

PROOF. The ideal $(I, \overline{H}: \overline{\sigma})$ certainly contains $I + (\overline{H}: \overline{\sigma})$ which by Corollary 8.2 is equal to $I + I(i_1, \ldots, i_{n-r})$ where $\sigma = (1, \ldots, n - r | i_1, \ldots, i_{n-r})$. However this ideal is an ideal in the second principal radical system of §4 and is prime by Theorem 4.1. The last statement follows as $\overline{\sigma}$ is not in the ideal.

Recall if H is a finite poset and $\{x_\alpha\} \subseteq R$ is a set of elements in a commutative ring R indexed by H, we say an ideal I of R is an H ideal if $J = \overline{I}$ for some ideal I of H. If J is an ideal we set J^* equal to the ideal generated by all $x_\tau \in J$.

DEFINITION 8.1 [28]. (See the introduction.) Let R, H, $\{x_\alpha\}$ be as above. The $\{x_\alpha\}$ are said to be a weak d-sequence if whenever I is an H ideal containing all x_β where $\beta < \alpha$ but not containing x_α then

(1) $(I: x_\alpha) \cap \overline{H} = (I: x_\alpha)^*$ is an H ideal.
(2) If $x_\alpha \in (I: x_\alpha)$ then $x_\alpha x_\beta \in I\overline{H}$.
(3) If $x_\alpha \notin (I: x_\alpha)$ then $(I: x_\alpha) = (I: x_\alpha^2)$.

THEOREM 8.1. *Let X, Y be generic matrices as above and let $R = Z[X, Y]/J$ where Z is the integers and J is the ideal generated by all the entries of XY and $I_{n-r}(Y)$. Let H be the poset of maximal minors of the top $n - r$ rows of Y. If $\alpha \in H$, let $\overline{\alpha}$ be the corresponding minor. Let H^+ be $H \cup \{0\}$ and let $\overline{0} =$ the $r \times r$ minor Δ of X determined by the last r columns of X. (Here $0 < \alpha$ for all $\alpha \in H$.) Then $\{\overline{\alpha} | \alpha \in H^+\}$ is a weak d-sequence.*

PROOF. Let I be as in Definition 8.1. Property (3) is immediate, since each $(I: \overline{\alpha})$ is prime by Lemma 8.3 and $\overline{\alpha} \notin (I: \overline{\alpha})$ so that $(I: \overline{\alpha}) = (I: \overline{\alpha}^2)$. If $\overline{\beta} \in (I: \overline{\alpha})$ then again by Lemma 8.3, $\overline{\beta} \not> \overline{\alpha}$ and then the standard quadratic relation (see [12], [13], or [25]) shows $\overline{\beta}\overline{\alpha} \in I\overline{H}$.

Finally, $(I: \overline{\alpha}) \cap \overline{H}^+ = \langle \overline{\beta} | \beta \not> \alpha \rangle + I = (I: \overline{\alpha})$ and this is an H ideal since $\langle \beta | \beta \not> \alpha \rangle$ is an ideal.

Recall if $\{x_\alpha\}$ is weak d-sequence an ideal J is said to be related to $\{x_\alpha\}$ if J is of the form $(I: x_\alpha) + \overline{H}$ where I is an H ideal containing all x_β with $\beta < \alpha$ but not x_α. The related ideals of the weak d-sequence above are given by Lemma 8.3; by Theorem 6.1 all of them are perfect and prime. Let $\alpha = (1, \ldots, n - r | s - n + r, \ldots, s)$, and let I be the ideal generated by all $\overline{\beta}$, $\beta \in H^+ - \{\alpha\}$. Then by Lemma 8.3, $(I: \overline{\alpha})$ is the ideal defined in R by the vanishing of all y_{ij} with $j < s + r - n$. This ideal clearly contains all other related ideals. We recall, (see the introduction)

THEOREM 8.2 [28]. *Suppose R is a commutative ring and $\{x_\alpha | \alpha \in H\}$ is a weak d-sequence. Suppose the ideal I generated by all the x_α is prime and $I_q^{(n)} = I_q^n$ for all primes q associated to any related ideal. Then $I^{(n)} = I^n$.*

THEOREM 8.3. *Let R be as in Theorem 8.1 and let $p = \langle \bar{\alpha} | \alpha \in H^+ \rangle$. Then $\mathrm{gr}_p(R)$ is a domain and consequently the ring $R[g_\tau / f]$ described above is integrally closed.*

PROOF. The last statement follows from Proposition 8.2 and the first statement. As p is prime (see [26]), to show $\mathrm{gr}_p(R)$ is a domain it is enough to show R_p is regular and $p^{(n)} = p^n$ for all n.

Let Q be the maximal related ideal defined by all y_{ij} with $j < s + r - n$ plus $(1, \ldots, n - r | r + s - n, \ldots, s)$ and Δ. Since Q is prime, if we show R_Q is regular and p_Q to be a complete intersection then $p_Q^{(n)} = p_Q^n$ for all n. But any other prime associated to a related ideal is equal to a related ideal (all related ideals are prime) and hence contained in Q. Thus if Q' is related $p_{Q'}^{(n)} = p_{Q'}^n$ and Theorem 8.2 shows that $p^{(n)} = p^n$ for all n. However, $p^{(n)} = p^n$ for all n and R_p regular imply $\mathrm{gr}_p(R)$ is a domain.

Let us consider R_Q. The ideal J defining R are the equations $XY = 0$ and $J_{n-r}(Y) = 0$. It is easy to see that any r by r minor of X is in the singular locus of $Z[X, Y]/J = R$. Certainly some $r \times r$ minor of X is not in Q, say $w = (1, \ldots, r | 1, \ldots, r)$ and so R_Q is regular. When we invert w, the ideal p is generated by Δ since the relation $f_\sigma g_{\bar{\sigma}, \tau} = f_{\bar{\sigma}} g_{\sigma, \tau}$ shows that for all τ, if we invert some $f_{\bar{\sigma}}$, then $\langle g_{\sigma, \tau} \rangle \subseteq (f_\sigma)$. Hence in R_Q, p_Q becomes a complete intersection and this shows $p_Q^{(n)} = p_Q^n$ for all n. Thus $p^{(n)} = p^n$ for all n as required.

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